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Boundary layer solutions to the Green's function for the extended Graetz problem

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Abstract—Limiting solutions are developed for the Green's function of the extended Graetz problem of mass (or heat) transfer, satisfying absorption boundary conditions, in flat channels. A differential equation directly satisfied by the Green's function of the transmission flux is derived. For finite Peclet numbers, the non-existence of similarity solutions is pointed out. Both uniform and parabolic velocity profiles are considered. While all orders of boundary layer solutions are obtained for the former case, only leading terms are derived for the latter, based on the approximation of a linearized profile near the wall. It is concluded that for sources confined close to the absorbing boundary, the axial diffusion effects cannot be ignored even at relatively large Peclet numbers.

INTRODUCTION

THE CLASSICAL Graetz problem, along with its extension to include axial diffusion (or conduction), is of interest in several mass (or heat) transfer applications in channel flows and has been treated extensively by several investigators [1–5]. Many practical applications involve the calculation of the Nusselt number and the penetration fraction for varying source distributions inside the channels, as well. The analytical solutions to these problems are generally presented either in the form of eigenfunction (Graetz type) series or in the form of limiting (Lévêque type) solutions. While the former type of solutions converge rapidly for large axial parameters measured from the point of discontinuity in the source, the latter do so for small axial parameters. In this paper, the Lévêque type solutions are investigated for mass transfer with axial diffusion (i.e. finite Peclet number, *Pe*), from a basic point of view. With the exception of Newman's [6] work, little appears to be available on this. However, Newman's extension is addressed to a particular type of source, namely uniform inlet distribution. It has an unsatisfactory [7] feature of adding the axial diffusion term to the mass balance equation and treating it as an initial value problem, in semi-infinite channels. Besides, the usual procedure of expanding the distribution function in terms of similarity variables, is not applicable for finite *Pe*. This is because when *Pe* is finite, the differential equation is not invariant under a stretching group of transformations, and therefore, it does not have similarity solutions. Moreover, the traditional method cannot be extended for upstream transport which is an important consequence of axial diffusion. Apart from this, the conventional assumption that at large Peclet numbers, the effect of axial diffusion on the parameters of downstream transport is perturbatively small, is not universally valid. In fact, when the sources are confined to the diffusion boundary layer, a significant upstream mass transfer will occur even at high Peclet numbers, thereby substantially reducing the fraction available for downstream transport. Such boundary layer sources arise in connection with the sampling of recoil atoms emitted following the radioactive decay of their parent atoms initially deposited on the channel walls [8]. In order to analyse this effect quantitatively, the point source response of the extended Graetz problem is investigated in the limiting region, via Green's function and Fourier transform techniques, both for uniform and parabolic velocity profiles.

FORMULATION

The problem for rectangular channels is formulated. Its extension to cylindrical ducts is rather straightforward. Let H be the half width between the channel walls, which are assumed to be infinite in extent. Let the X–Z plane coincide with one of the walls, with $x \in (-\infty, \infty)$, and the Y-axis be perpendicular to them $\{y \in [0, 2H]\}$. The fluid is assumed to flow in the positive direction of the X-axis with a centre-line velocity v_m. It is required to study the response of this channel (with absorbing walls) to a line source placed at x_0, y_0 perpendicular to the direction of flow and parallel to the walls. The Peclet number and the non-dimensional coordinates are defined as

$$Pe \equiv v_m H/D$$

where D is the diffusion coefficient of the particles

$$\eta \equiv y/H \in [0, 2]$$

and

$$\mu \equiv (\tilde{x}/Pe)$$

where

NOMENCLATURE

$Ai(x)$	Airy function of the first kind
a_n	$Pe(2n+\eta)$
b_n	$Pe(2n+2-\eta)$
c	real number $(0 < c < 1)$
D	diffusion coefficient of particles
$F(x)$	integral defined in equation (A3)
$g(\eta, \mu/\eta_0, \mu_0)$	Green's function for the concentration field
H	lateral half width of the channel
$J^{\pm}(\eta, \mu)$	cumulative deposition fluxes
$K_0(x), K_1(x)$	modified Bessel functions of orders 0 and 1
Pe	Peclet number, $v_m H/D$
p, q	complex integration variables
$U_j(u, v)$	function defined in equation (10b) for $j = 0, 1$
$v(\eta)$	1 for plug and $(2\eta-\eta^2)$ for parabolic velocity profiles
v_m	centre-line velocity of the stream
x	X (axial) coordinate
$\tilde{x}$	$x/H \in (-\infty, \infty)$
y	Y (lateral) coordinate $\in [0, 2H]$.

Greek symbols

$\delta(\mu)$	Dirac delta function
ζ	$\xi^{4/3}$
η	reduced lateral distance $(y/H) \in [0, 2]$
η_0	reduced lateral distance for the source
$\theta(\eta, \mu)$	Green's functions for the transmission flux
μ	reduced axial distance, $Dx/v_m H^2$
μ_0	reduced axial distance for the source
v	axial variable for plug profile, $Pe\,\tilde{x}$
ξ	lateral variable for parabolic profile, $\eta(2Pe)^{1/2}$
ρ	real integration variable
τ	axial variable for parabolic profile, $\tilde{x}(2Pe)^{1/2}$
$\phi(\eta, \omega)$	solution of equation (6a)
ω	transform variable conjugate to μ.

$$\tilde{x} \equiv (x/H) \in (-\infty, \infty).$$

The Green's function $g(\eta, \mu/\eta_0, \mu_0)$ for the concentration distribution satisfies the equation

$$\frac{\partial^2 g}{\partial \eta^2} + \frac{1}{Pe^2}\frac{\partial^2 g}{\partial \mu^2} - v(\eta)\frac{\partial g}{\partial \mu} + \delta(\mu-\mu_0)\delta(\eta-\eta_0) = 0 \tag{1a}$$

where $v(\eta)$ (1 for plug flow, $2\eta-\eta^2$ for the parabolic profile) is the velocity profile and δ the Dirac delta function. The boundary conditions are

$$g(0, \mu/\eta_0, \mu_0) = g(2, \mu/\eta_0, \mu_0) = g(\eta, \pm\infty/\eta_0, \mu_0) = 0. \tag{1b}$$

In most of the situations, the quantity of practical interest is the Green's function for the transmission flux, which is defined by

$$\theta(\mu/\eta_0, \mu_0) \equiv \int_0^2 [v(\eta) - Pe^{-2}(\partial/\partial\mu)]g(\eta, \mu/\eta_0, \mu_0)\,\mathrm{d}\eta, \quad \mu \in (-\infty, \infty). \tag{2}$$

A differential equation (in the source coordinates, η_0 and $\mu-\mu_0$) for $\theta(\mu/\eta_0, \mu_0)$ may be easily constructed from equation (1a) via its adjoint equation by the well-known procedure [9]. Upon setting $\mu_0 = 0$, dropping the suffix from η_0, and redefining $\theta \equiv \theta(\eta, \mu)$, one obtains

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{Pe^2}\frac{\partial^2 \theta}{\partial \mu^2} - v(\eta)\frac{\partial \theta}{\partial \mu} + v(\eta)\delta(\mu) - (1/Pe^2)\delta'(\mu) = 0, \quad \eta \in [0, 2], \quad \mu \in (-\infty, \infty) \tag{3a}$$

along with

$$\theta(0, \mu) = \theta(2, \mu) = \theta(\eta, \pm\infty) = 0. \tag{3b}$$

It follows from equation (3a) that

$$\theta(\eta, 0^+) - \theta(\eta, 0^-) = 1 \text{ and } \frac{\partial\theta}{\partial\mu}(\eta, 0^+) = \frac{\partial\theta}{\partial\mu}(\eta, 0^-). \tag{3c}$$

Besides, $\theta(\eta, \mu)$ is negative when $\mu < 0$. The Nusselt number J^+, which is the total quantity deposited up to a length μ (say $\mu > 0$) is given by

$$J^+(\eta, \mu) = \theta(\eta, 0^+) - \theta(\eta, \mu), \quad \mu > 0. \tag{4}$$

Therefore, the knowledge of $\theta(\eta, 0^+)$—which will be substantially different from unity for boundary layer sources due to upstream transmission—is essential for the calculation of the deposition flux. It should be remarked here that when $Pe = \infty$, equation (3a), along with equation (3b) is completely equivalent to the classical Graetz problem. Hence its limiting solutions are classical Lévêque solutions. Similarly, equation (3a) can be shown to be equivalent to a heat transfer problem [10] with a step change in temperature at $\mu = 0$ on the channel walls. However, this equation cannot be split up into a self contained set of one-sided equations separately for $\mu < 0$ and $\mu > 0$, thereby making the Newman procedure [6] of similarity expansion inapplicable. On the other hand, the well-known asymptotic methods in Fourier inversion offer powerful tools for this purpose. Both uniform and parabolic velocity profiles will be considered in the next section. While the latter case is more important from a practical point of view, the former corresponds to certain moving source problems [11] and is mathematically simpler.

LIMITING SOLUTIONS

The upstream mass transfer due to a steady source is not solely due to axial diffusion, but arises as a consequence of the combined effect of axial diffusion and the absorption boundary conditions at the walls. In fact, in the absence of absorption, the drift and the diffusion currents are balanced in the upstream region due to statistical equilibrium. The above statement also implies that when the sources are far removed from the boundaries (i.e. when the absorption is small), the upstream transmission vanishes asymptotically. In the limit of small μ, therefore, a significant contribution to θ for $\mu < 0$ (or to $1-\theta$ for $\mu > 0$) arises only if the source is close to one of the boundaries. Thus, the limiting solutions are also boundary layer solutions. All orders of such solutions will be obtained for plug flow; for a parabolic profile, those beyond the linearized (i.e. Couette flow) regime are difficult to obtain.

Now, upon using the method of Fourier transforms with respect to the variable μ, the formal solution to equation (3a) may be written as

$$\theta(\eta,\mu) = \frac{\mu}{2|\mu|} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp(-i\omega\mu)\phi(\eta,\omega)\frac{d\omega}{\omega}, \quad \mu\in(-\infty,\infty) \quad (5)$$

where $\phi(\eta,\omega)$ satisfies the following differential equation with respect to η:

$$\phi''(\eta,\omega) + i\omega[v(\eta) + (i\omega/Pe^2)]\phi(\eta,\omega) = 0, \quad \eta\in[0,2] \quad (6a)$$

along with

$$\phi(0,\omega) = \phi(2,\omega) = 1. \quad (6b)$$

When the fluid is stagnant, i.e. $v(\eta) = 0$ (or $Pe \to 0$), the solutions of equation (6a) are linear combinations of $\exp[\pm|\omega|\eta/Pe]$. This, along with boundary conditions (6b) reduces equation (5) to

$$\theta(\eta,\tilde{x}) = (1/\pi)\tan^{-1}\left[\frac{\sin(\pi\eta/2)}{\sinh(\pi\tilde{x}/2)}\right] \xrightarrow[\eta,\tilde{x}\to 0]{} (1/\pi)\tan^{-1}[\eta/\tilde{x}], \quad Pe = 0. \quad (7)$$

This is an exact result. Its importance lies in the fact that, irrespective of the form of the velocity profile, all boundary layer solutions corresponding to finite Pe, approach the second form of equation (7) in the limit $\eta, \tilde{x} \to 0$. This may be easily seen by noting that, the form of $\theta(\eta,\mu)$ as $\tilde{x}$ (or μ) $\to 0$, is governed by the form of $\phi(\eta,\omega)$ as $|\omega| \to \infty$. When $\eta \to 0$, the asymptotic ($|\omega| \to \infty$) solution to equation (6a) is $\exp(-|\omega|\eta/Pe)$, irrespective of the form of $v(\eta)$ so long as it is bounded, and this readily integrates in equation (5) to yield the second form of equation (7).

Plug flow: $v(\eta) = 1$

In this case, equation (6a) has the solution

$$\phi(\eta,\omega) = \frac{\cos[(1-\eta)(i\omega)^{1/2}(1+i\omega/Pe^2)^{1/2}]}{\cos[(i\omega)^{1/2}(1+i\omega/Pe^2)^{1/2}]}. \quad (8)$$

Upon transforming, $i\omega \equiv -Pe^2 q$ and $\nu \equiv Pe^2\mu = Pe\ \tilde{x}\in(-\infty,\infty)$ and substituting equation (8) in equation (5), one has

$$\theta(\eta,\nu) = \frac{1}{2}\left(1+\frac{\nu}{|\nu|}\right) - \frac{1}{2\pi i} \times \int_{c-i\infty}^{c+i\infty}\left[\frac{\exp(\nu q)\cosh\{Pe(1-\eta)q^{1/2}(1-q)^{1/2}\}}{q\cosh\{Pe\,q^{1/2}(1-q)^{1/2}\}}\right]dq, \quad -\infty < \nu < \infty \text{ and } 0 < c < 1. \quad (9)$$

As shown in the Appendix, equation (9) may be simplified as follows:

$$\theta(\eta,\mu) = e^{\nu/2}\sum_{n=0}^{\infty}(-1)^n[U_0(a_n,\nu) + U_0(b_n,\nu) + \nu\{U_1(a_n,\nu) + U_1(b_n,\nu)\}], \quad \nu\in(-\infty,\infty) \quad (10a)$$

where $a_n \equiv Pe(2n+\eta)$, $b_n \equiv Pe(2n+2-\eta)$ and

$$U_j(z,\nu) \equiv \int_0^z [t^2+\nu^2]^{-j/2}K_j[\tfrac{1}{2}(t^2+\nu^2)^{1/2}]\,dt, \quad j = 0, 1. \quad (10b)$$

(K_0 and K_1 are the modified Bessel functions.) Besides $U_j(z,-\nu) = U_j(z,\nu)$ and [12a)

$$U_j(\infty,|\nu|) = \pi|\nu|^{-j}\exp(-|\nu|/2), \quad j = 0, 1. \quad (10c)$$

The successive terms of equation (10a) represent the higher order solutions arising out of the reflection effects from the opposite wall. Either when $|\nu| \approx 0$ for a given η or when $\eta \approx 0$ for a given ν, it may be shown from equations (10a)–(10c) that the leading approximation is

$$\theta(\eta,\pm|\nu|) \approx \frac{1}{2} \pm \frac{1}{2} - \frac{e^{\nu/2}}{2\pi}\int_{Pe\eta}^{\infty}\left[K_0\{\tfrac{1}{2}\surd(t^2+\nu^2)\} + \frac{\nu}{\surd(t^2+\nu^2)}K_1\{\tfrac{1}{2}\surd(t^2+\nu^2)\}\right]dt. \quad (11)$$

Equation (11) is an exact result for the case of mass transfer over a single plate ($\eta = 0$) as well. It also satisfies all the requirements (equations (3b) and (3c)) including those specific to the boundary layer solutions as mentioned at the beginning of this section. Moreover, the quantities, $\theta(\eta,\pm 0)$, which are required for the computation of J may be easily evaluated from equation (11) by setting $|\nu| = 0$. From this, it follows that $\theta(\eta,\pm 0) \to \pm 1/2$, as $\eta \to 0$.

As $Pe \to \infty$, it may be verified from the well-known asymptotic expansions for K_0 and K_1 that, for $\mu > 0$, equation (10a) goes over the usual error function series in terms of the similarity variable, $\eta/2\surd\mu$. On the other hand, when $\mu < 0$, it may be shown that

$$\theta(\eta,\nu) \underset{|\nu|\to\infty}{\sim} -(1/2\surd\pi)Pe\,\eta|\nu|^{-3/2} \times \exp[\nu + Pe^2\eta^2/(4\nu)], \quad \nu < 0. \quad (12a)$$

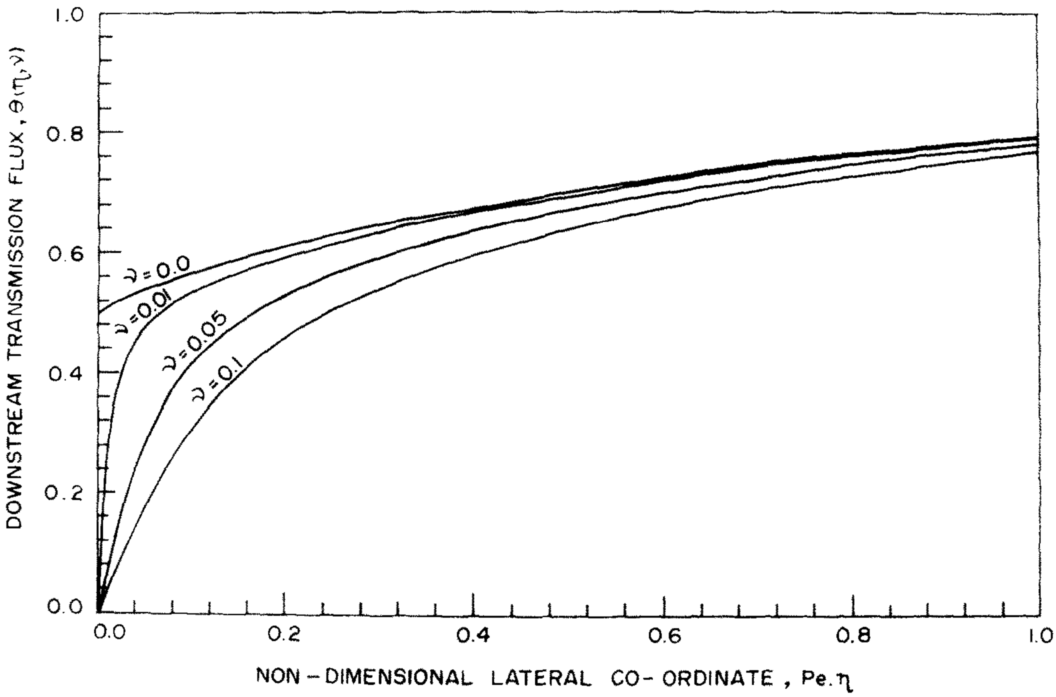

FIG. 1. Variation of the downstream transmission flux [$\theta(\eta, v)$, $v > 0$] in flat ducts with respect to the lateral boundary parameter, $Pe\,\eta$, for a few limiting values of the dimensionless axial parameter v ($\equiv Pe\,\tilde{x}$) in the case of a uniform velocity profile (equation (11)). (Pe = Peclet number, η and $\tilde{x}$ are respectively the lateral and the axial distances measured in terms of the channel half width.)

The solutions represented by equation (11) are plotted in Figs. 1 and 2 separately both for $v > 0$ and $v < 0$. It may be seen from Fig. 2 that the magnitude of the upstream transmission flux possesses a maximum with respect to η, which gradually shifts to a value of 0.5 at $\eta = 0$ and $v = 0^-$. This maximum arises due to the interplay of the two competing processes, namely the statistical equilibrium when the source is far away and the absorption condition when the source is near the wall. Approximately, this maximum can be shown (consistent with the approximations involved in deriving equation (12a)) to occur at

$$\eta_{\max} \sim (2|\mu|)^{1/2}, \quad \mu < 0. \tag{12b}$$

It must be mentioned in passing that the series representation (equation (10a)) for $\theta(\eta, v)$ can also be converted into a real integral representation, which reduces to equation (7) for $Pe = 0$. However, for $Pe = \infty$, such a real integral representation does not exist.

Parabolic profile: $v(\eta) = 2\eta - \eta^2$

For this case, the solutions to equation (6a) are the parabolic cylinder functions. If one develops a perturbation around the centre-line ($\eta = 1$) using the asymptotics of Olver [13], it may be shown that the lowest order solution is identical to that for plug flow, i.e. equation (10a). While any order of perturbative expansion may be developed this way for sources close to the centre-line, it does not lead to a boundary layer solution which will be superior to the ubiquitous result of equation (7). One can of course use somewhat involved expansions of Olver for $\phi(\eta, \omega)$ around $\eta = 0$; however, a more direct way to obtain leading solutions would be to approximate the parabolic velocity profile by a linear one in the boundary layer region.

In this spirit, one sets, $2\eta - \eta^2 \approx 2\eta$ ($\eta \approx 0$) and relaxes the domain of η as $\eta \in [0, \infty)$, in equation (6a). Upon substituting $q \equiv -i\omega/Pe^2$ and seeking the solution of equation (6a) which tends to unity as $\eta \to 0$ and vanishes as $\eta \to \infty$ ($q \neq 0$), one has

$$\phi(\eta, q) = \frac{Ai[(2Pe^2\,q)^{1/3}(\eta - q/2)]}{Ai[-(Pe/2)^{2/3}(q)^{4/3}]}, \qquad \eta \geqslant 0 \text{ and } q = i\omega/Pe^2. \tag{13}$$

In the complex q-plane, $\phi(\eta, q)$ has a branch point at $q = 0$ with the branch cut extending along the negative real axis, and a series of simple poles along the positive real axis, i.e. the complete set of eigenvalues has both discrete and continuous parts. Correspondingly, the downstream ($\mu > 0$) flux will be given by an integral along the branch line and the upstream ($\mu < 0$) flux will be a series of residues evaluated at the poles. This is unlike the plug flow case in which, in the boundary layer approximation, the branch cut extends on a part of the positive real axis as well (Appendix).

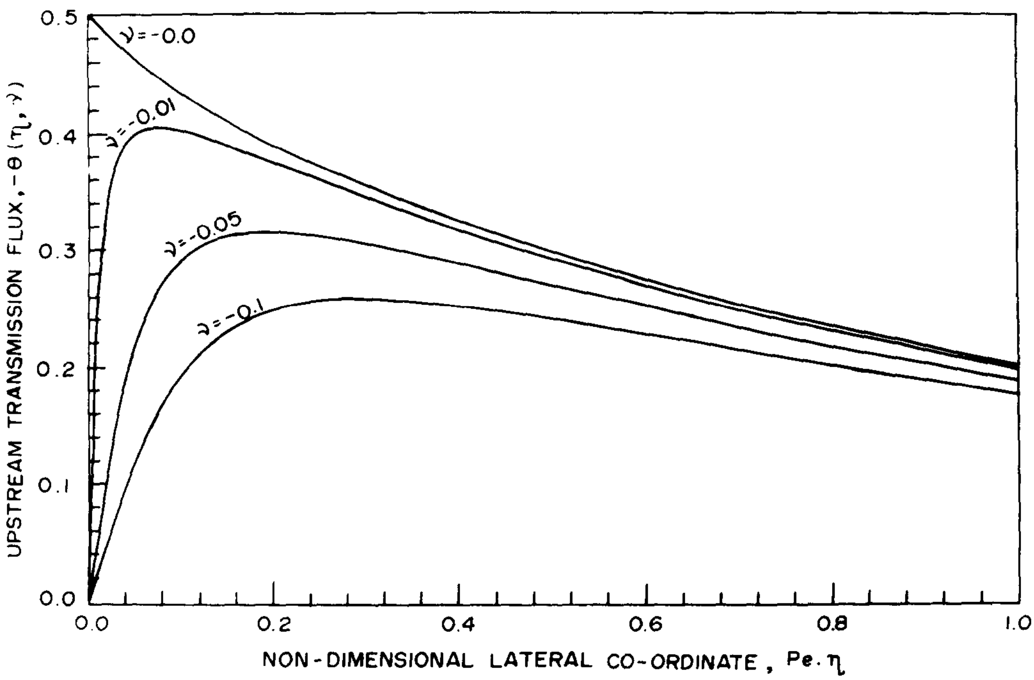

FIG. 2. Variation of the upstream transmission flux $[-\theta(\eta, \nu), \nu < 0]$ in flat ducts with respect to the lateral boundary layer parameter, $Pe\,\eta$, for a few limiting values of the dimensionless axial parameter ν ($\equiv Pe\,\tilde{x}$) in the case of a uniform velocity profile (equation (11)). (Pe = Peclet number, η and $\tilde{x}$ are respectively the lateral and the axial distances measured in terms of the channel half width.)

Case (i): $\mu > 0$. In view of the remarks made above, one may reduce equation (5) using equation (13) to the following real integral for the downstream flux, after some rearrangement:

$$\theta(\xi, \tau) = \frac{3}{\pi} \int_0^\infty \exp(-\tau\xi u^3)[[Ai[-\zeta(u + u^4)]Bi[-\zeta u^4] - Bi[-\zeta(u+u^4)]Ai[-\zeta u^4]]/ [Ai^2[-\zeta u^4] + Bi^2[-\zeta u^4]]]\frac{du}{u}, \quad \tau > 0 \quad (14)$$

where, τ, ξ (or ζ) are the boundary layer variables for Couette flow, defined by

$$\xi \equiv (2Pe)^{1/2}\eta, \; \zeta \equiv \xi^{4/3} \text{ and } \tau \equiv (2Pe^3)^{1/2}\mu = \tilde{x}(2Pe)^{1/2}. \quad (15)$$

When $Pe = \infty$, equation (14) can be shown to reduce to the well-known classical Lévêque limit in the similarity variable, $\eta\mu^{-1/3}$. Figure 3 shows the variation of $\theta(\xi, \tau)$ (equation (14)) as a function of ξ for a few values of τ.

Case (ii): $\mu < 0$. The poles of $\phi(\eta, q)$ in equation (13), occur at $q_n = (2/Pe)^{1/2} r_n^{3/4}$, where $-r_n$ ($n = 0, 1, \ldots$) are the zeros of the Airy function Ai. The residues of equation (5) may be easily evaluated at these poles to yield

$$\theta(\xi, \tau) = -\frac{3}{4}\sum_{n=0}^{\infty} \exp[\tau r_n^{3/4}] \frac{Ai[-r_n + \xi r_n^{1/4}]}{r_n\, Ai'[-r_n]}, \quad \tau < 0. \quad (16)$$

When $|\tau|$ is small, a large number of terms would be required in the above series. For large n, $r_n \sim (3\pi/2)^{2/3}(n + 3/4)^{2/3}$. Figure 4 shows the variation of $|\theta(\xi, \tau)|$ as a function of ξ, for a few values of τ. As in the case of plug flow, here too, the upstream flux possesses a maximum which tends to 0.5 at $\xi = \tau = 0$.

For large Pe and small $|\tau|$, instead of equation (16), a simpler approximation may be derived in the following way (this applies to downstream flux as well). Since, q ($\equiv -i\omega/Pe^2$) occurring in equation (13) is essentially an integration variable in equation (5), it may be transformed to $p \equiv 1 - q/2\eta$, for $\tau < 0$. Once expressed in terms of p, equation (13) may be reduced, by a careful use of asymptotics for the Airy functions, to

$$\phi(\xi, p) \sim \exp[-\xi^2 p^{1/2}(1-p)^{1/2}].$$

This is of the same form as that for plug flow (Appendix), and hence

$$\theta(\xi, \tau) \sim (1/2\pi)\exp[\xi\tau/2][\xi\tau U_1(\xi^2, \xi\tau) + U_0(\xi^2, \xi\tau)] \quad (17)$$

where the functions U_0 and U_1 are defined in equation (10b). Incidentally, the same expression is valid even for $\tau > 0$, under similar approximations. Comparison

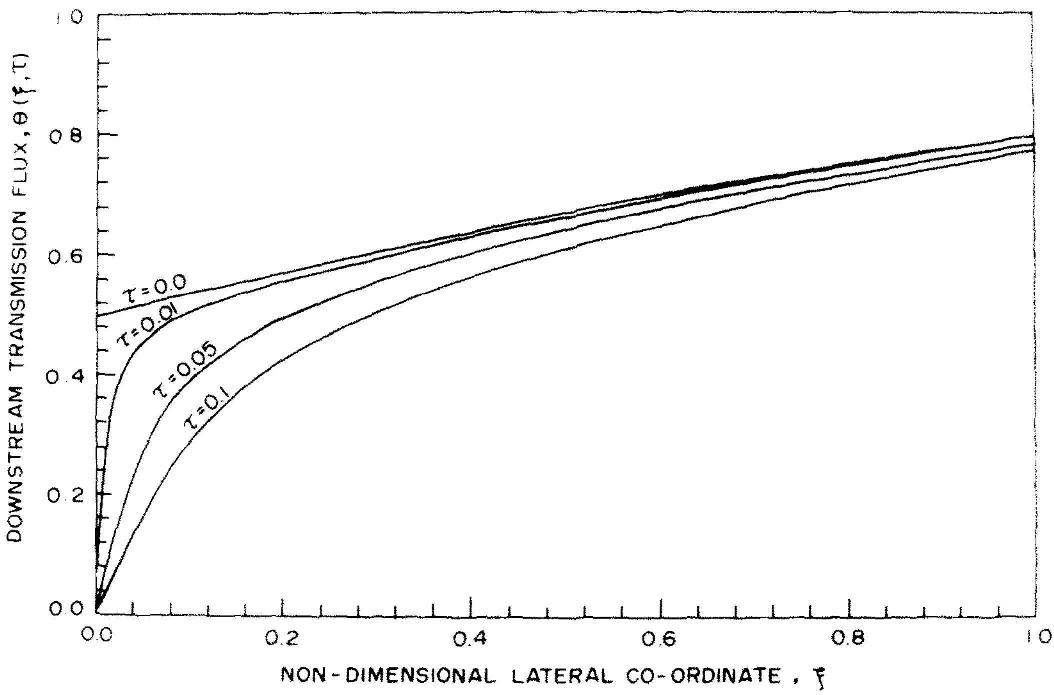

FIG. 3. Variation of the downstream transmission flux $[\theta(\xi, \tau), \tau > 0]$ in flat ducts with respect to the lateral boundary layer parameter, ξ $[\equiv \eta(2Pe)^{1/2}]$, for a few limiting values of the dimensionless axial parameter τ $[\equiv \bar{x}(2Pe)^{1/2}]$ in the case of a parabolic velocity profile (equation (14)). (Pe = Peclet number, η and $\bar{x}$ are respectively the lateral and the axial distances measured in terms of the channel half width.)

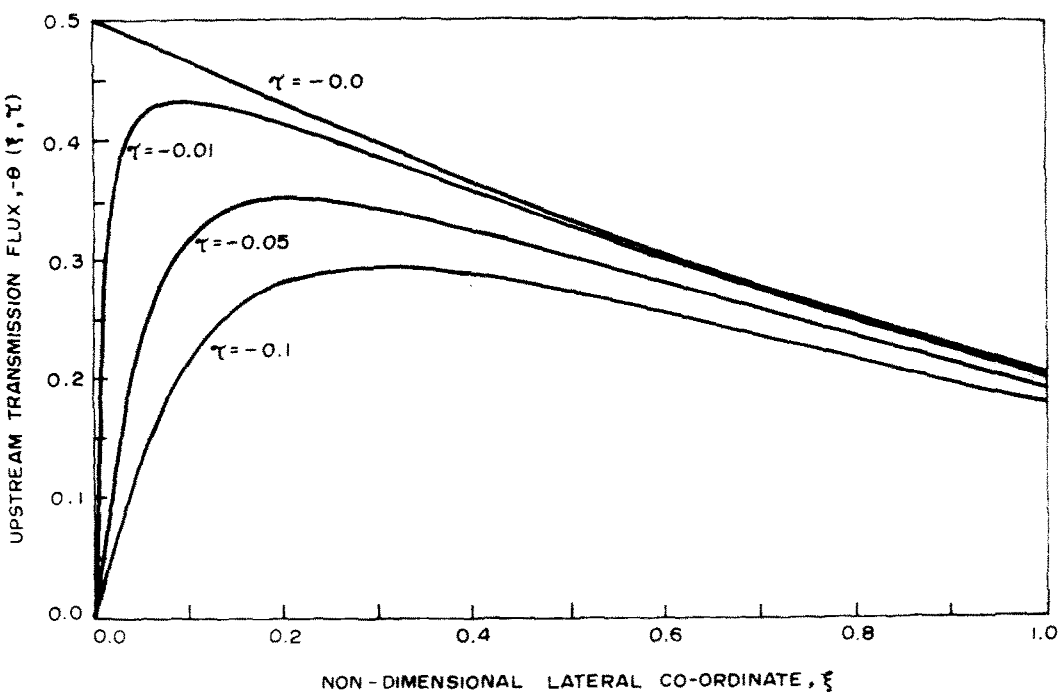

FIG. 4. Variation of the upstream transmission flux $[-\theta(\xi, \tau), \tau < 0]$ in flat ducts with respect to the lateral boundary layer parameter, ξ $[\equiv \eta(2Pe)^{1/2}]$, for a few limiting values of the dimensionless axial parameter τ $[\equiv \bar{x}(2Pe)^{1/2}]$ in the case of a parabolic velocity profile (equation (16)). (Pe = Peclet number, η and $\bar{x}$ are respectively the lateral and the axial distances measured in terms of the channel half width.)

of these with the full equation (equation (14) or (16)) suggests a good agreement between the two in the boundary layer region ($\xi \approx 0$).

For $\tau < 0$, using equation (17), the maximum in the upstream flux can be shown to occur at

$$\xi_{\max} \underset{\tau \sim 0^-}{\sim} (-\tau/3)^{1/3} \quad \tau < 0. \tag{18}$$

In reality, however, under Couette flow approximations, the point of maximum does not indefinitely increase with respect to $|\tau|$ as predicted by the above. As $\tau \to -\infty$, it is limited by the first term in the series (equation (16)), i.e. $\xi_{\max} \to 1.0167$ as $\tau \to -\infty$.

CONCLUSIONS

In this analysis, starting from the differential equation directly satisfied by Green's function for the transmission flux, one has been able to obtain all orders of boundary layer solutions for a uniform velocity profile and only leading terms for a parabolic profile. As can be seen from this, the upstream flux is nearly 0.5 for boundary layer sources for both the velocity profiles, thereby substantiating our earlier assertion regarding the dominating effect of axial diffusion in such cases. Therefore, for realistic boundary layer sources extending in the positive X-direction (say) a significant wall deposition in the entrance region would be due to upstream diffusion effects. For general sources, a detailed calculation of the wall deposition must take into account both the contributions from all source points.

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APPENDIX. EVALUATION OF THE INVERSION INTEGRAL (EQUATION (9))

Upon using the identity

$$[\cosh z]^{-1} \equiv 2 \sum_{n=0}^{\infty} (-1)^n \exp[-(2n+1)z], \quad z > 0 \tag{A1}$$

equation (9) may be rewritten as

$$\theta(\eta, v) = \frac{1}{2}\left(1+\frac{v}{|v|}\right) - \sum_{n=0}^{\infty} (-1)^n [F(a_n)+F(b_n)] \tag{A2}$$

where, $a_n \equiv (2n+\eta)Pe$, $b_n \equiv (2n+2-\eta)Pe$ and

$$F(x) \equiv (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \exp[vq - xq^{1/2}(1-q)^{1/2}]q^{-1}\,dq,$$

$$v > 0 \text{ and } 0 < c < 1. \tag{A3}$$

The integrand in equation (A3) has branch points at $q = 0$ and 1. The corresponding branch lines are between $[0, -\infty)$ and $[1, \infty)$. For $v > 0$, upon constructing a contour on the negative q-plane, equation (A3) may be reduced to the following integral over the real line:

$$F(x) = 1-(\pi)^{-1} \int_0^{\infty} e^{-v\rho} \sin[x\rho^{1/2}(1+\rho)^{1/2}]\rho^{-1}\,d\rho, \quad v > 0. \tag{A4}$$

Upon transforming the integration variable to $t \equiv 2\rho^{1/2}(1+\rho)^{1/2}$ and noting the following identities, namely [12b]:

$$K_0[(a^2+b^2)^{1/2}] = \int_0^{\infty} (1+t^2)^{-1/2} \exp[-b(1+t^2)^{1/2}] \cos(at)\,dt \tag{A5}$$

and

$$K_1(u) \equiv -K_0'(u) \tag{A6}$$

one arrives at

$$F(x) = 1 - \frac{e^{v/2}}{2\pi} \int_0^x [v(t^2+v^2)^{-1/2} K_1\{\tfrac{1}{2}\sqrt{(t^2+v^2)}\} + K_0\{\tfrac{1}{2}\sqrt{(t^2+v^2)}\}]\,dt, \quad v > 0. \tag{A7}$$

Upon substituting equation (A7) in equation (A2) and on using equation (10b) and the identity, $\sum_{n=0}^{\infty}(-1)^n = 1/2$, one arrives at equation (10a) valid for $v > 0$.

For $v < 0$, upon transforming the integration variable in equation (A3) to $p \equiv 1-q$, and following the same procedure as above, one arrives once again at equation (10a).

SOLUTIONS DE COUCHE LIMITE AVEC FONCTION DE GREEN POUR LE PROBLEME GENERAL DE GRAETZ

Résumé—Des solutions limites sont développées pour la fonction de Green dans le problème de Graetz du transfert de masse (ou de chaleur), satisfaisant les conditions aux limites d'absorption dans les canaux plats. On dérive une équation différentielle directement satisfaite par la fonction de Green de la transmission du flux. Pour des nombres de Peclet finis, on dégage l'inexistence des solutions similaires. On considère les profils uniformes et paraboliques de vitesse. Alors que tous les ordres des solutions de couche limite sont obtenus pour le premier cas, seuls les termes principaux sont dérivés pour le dernier, à partir de l'approximation d'un profil linéarisé près de la paroi. On conclut que pour des sources confinées près de la frontière absorbante, les effets de la diffusion axiale ne peuvent être ignorés même pour des nombres de Peclet relativement grands.

GRENZSCHICHT-LÖSUNGEN DER GREEN'SCHEN FUNKTION FÜR DAS ERWEITERTE GRAETZ-PROBLEM

Zusammenfassung—Es werden Lösungen für Grenzfälle der Green'schen Funktion für das erweiterte Graetz-Problem des Stoff- (oder Wärme-)transports in flachen Kanälen entwickelt, die den Absorptionsrandbedingungen genügen. Es wird eine Differentialgleichung abgeleitet, der die Green'sche Funktion für die übertragene Stromdichte direkt genügt. Für endliche Peclet-Zahlen wird das Fehlen von Ähnlichkeits-Lösungen aufgezeigt. Es werden sowohl gleichförmige als auch parabolische Geschwindigkeitsprofile betrachtet. Während man im ersten Fall alle Ordnungen der Lösung der Grenzschichtgleichungen erhält, werden—basierend auf der Näherung linearer Profile in Wandnähe—im zweiten Fall nur die Hauptterme abgeleitet. Es wird gefolgert, daß die axiale Diffusion für Quellen nahe der Absorptionsgrenze selbst für relativ große Peclet-Zahlen nicht vernachlässigt werden kann.

РЕШЕНИЯ В ПРИБЛИЖЕНИИ ПОГРАНИЧНОГО СЛОЯ ОБОБЩЕННОЙ ЗАДАЧИ ГРЕТЦА С ПОМОЩЬЮ ФУНКЦИИ ГРИНА

Аннотация—Получены удовлетворяющие граничным условиям абсорбции предельные решения с помощью функции Грина обобщенной задачи Гретца о переносе массы (или тепла) в плоских каналах. Выведено дифференциальное уравнение для массопереноса, непосредственно решаемое с помощью функции Грина. Отмечено отсутствие автомодельных решений при конечных значениях числа Пекле. Рассмотрены как однородные по сечению, так и параболические профили скорости. В то время как в первом случае получены решения уравнений пограничного слоя всех порядков, в последнем определены только главные величины путем аппроксимации линеаризованного профиля у стенки. Сделан вывод о том, что в случае когда источники находятся вблизи поглощающей границы, нельзя пренебрегать эффектами осевой диффузии даже при относительно больших значениях числа Пекле.